LE MATEMATICHE
Vol. LIV (1999) – Fasc. II, pp. 243–259

FANO MANIFOLDS AS AMPLE DIVISORS

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We study polarized manifolds (X, L) with L having a smooth element A in its linear system which is a Fano manifold of coindex 3 and second Betti number greater or equal than 2.

1. Introduction.

Let A be a complex projective manifold of dimension $n \geq 4$ which is an ample divisor in a projective manifold X. Let $L = \mathcal{O}_x(A)$ be the line bundle on X associated to the divisor A. We are interested in the classification of polarized pairs (X, L) with $A \in |L|$ a Fano manifold of coindex 3. Such classification has been worked out in [19] under the assumption that $b_2(A) = 1$. It is natural to extend such classification to polarized pairs (X, L) with $A \in |L|$ a Fano manifold of coindex 3 and $b_2(A) \geq 2$. While the classification in the case $b_2(A) = 1$ is fairly straightforward, the one in which $b_2(A) \geq 2$ is more involved.

The main reason for being interested in such classification is the fact that among the Fano manifolds A of coindex 3 and $b_2(A) \geq 2$ there are manifolds with a $\mathbb{P}^1$-bundle structure either over $\mathbb{P}^3$ or $\mathbb{Q}^3$. These manifolds are natural candidates for examples supporting the standing conjecture on smooth $\mathbb{P}^d$-bundles, $p : A \to B$, over a manifold B of dimension b, as ample divisor, ([3], (5.5.1)).

Entrato in Redazione il 7 giugno 1999
Partially supported by MURST in the framework of the project: "Geometria algebrica, algebra commutativa e aspetti computazionali"

Such conjecture, which we recall in the last section of this paper, has been shown except when $d = 1$, $b \geq 3$ and when the base B does not map finite-to-one into its Albanese variety. The case when either $d \geq 2$ or B is a submanifold of an Abelian variety follows from Sommese's extension theorems, see [23] and [11]. The conjecture is also known in the cases $d = 1$ and $b \leq 2$. For $d = 1$ and $b = 1$ see [1] and [2], while for $d = 1$ and $b = 2$ see [8], [9], [22] and [21].

The paper is organized as follows. In section 2 we give the preliminaries and recall, for the convenience of the reader, the theorems needed in the paper. In section 3 we prove a general result about $\mathbb{P}^1$-bundles over a smooth projective 3-fold which will be needed later on in the paper. In section 4 we classify polarized pairs (X, L) with $A \in |L|$ a Fano manifold of coindex 3 and $b_2(A) \geq 2$. In the last section we make some final remarks.

2. Notations and Preliminaries.

In this section we recall some definitions and results which will be needed throughout the paper. The notation used is the standard one in adjunction theory (see [3], [10]).

We work over the complex field $\mathbb{C}$. By a manifold we mean a smooth projective variety over $\mathbb{C}$.

Line bundles and invertible sheaves of their sections are used with little or no distinction. Hence we will freely switch from the multiplicative to the additive notation and viceversa.

Definition 2.0.1. Let L be a line bundle on a manifold X. L is said to be *nef* if $L \cdot D \geq 0$ for all effective curves D on X, and in this case L is said to be *big* if $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L.

Definition 2.0.2. Let X be a complex projective manifold. Let K_X be the canonical divisor of X. We say that X is a Fano manifold if $-K_X$ is linearly equivalent to rH, where H is an ample divisor on X. If r is the largest integer dividing $-K_X$ then r is called the *index* of X. The integer $\dim X - r + 1$ is called *coindex* of X.

Fano manifolds of coindex 3 are well understood, see [17], [24] and [25] for dimension ≥ 4. We recall the structure of such Fano manifolds with $b_2 \geq 2$ since it will be used later on in section 4.

Theorem 2.1. ([12], [14]). *Let X be a Fano 4-fold of coindex 3 and of product type, i.e. $X \cong \mathbb{P}^1 \times M$ where M is a Fano 3-fold of even index. Then M is one* *of the following:* $\mathbb{P}^3$, V_d, W, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, *where* V_d *is a Del Pezzo manifold with* $d = 7$ *or* $1 \leq d \leq 5$ *and* W *is a divisor of bidegree* $(1, 1)$ *on* $\mathbb{P}^2 \times \mathbb{P}^2$.

Theorem 2.2. ([17], [24], [25]). *Let* X *be a Fano manifold of dimension* ≥ 4, *coindex 3,* $b_2 \geq 2$ *and with a smooth 3-dimensional section. If* X *is not a Fano 4-fold of product type, then* X *is isomorphic to one of the following or a linear section of its fundamental model:*

(i) *a double cover of* $\mathbb{P}^2 \times \mathbb{P}^2$ *whose branch locus is a divisor of bidegree* $(2, 2)$*;*
(ii) *a divisor of* $\mathbb{P}^2 \times \mathbb{P}^3$ *of bidegree* $(1, 2)$
(iii) $\mathbb{P}^3 \times \mathbb{P}^3$*;*
(iv) $\mathbb{P}^2 \times \mathbb{Q}^3$*;*
(v) *the blow up of a smooth 4-dimensional quadric* $\mathbb{Q}^4 \subset \mathbb{P}^5$ *along a conic* C *on it such that the plane* $< C >$ *spanned by* C *is not contained in* $\mathbb{Q}^4$*;*
(vi) *the blow up of* $\mathbb{P}^5$ *along a line;*
(vii) X *has two* $\mathbb{P}^1$*-bundle structures and can be realized either as* $P(NCB)$, *where* NCB *is the null correlation bundle over* $\mathbb{P}^3$*, that is a stable rank-2 bundle with* $c_1 = 0$, $c_2 = 1$*, or* $P(\mathcal{E})$*, where* $\mathcal{E}$ *is a stable rank-2 bundle on* $\mathbb{Q}^3$ *with* $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 1$*;*
(viii) *the* $\mathbb{P}^1$*-bundle* $P(\mathcal{O}_{\mathbb{Q}^3}(-1) \oplus \mathcal{O}_{\mathbb{Q}^3})$ *over* $\mathbb{Q}^3 \subset \mathbb{P}^4$*;*
(ix) *the* $\mathbb{P}^1$*-bundle* $P(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ *over* $\mathbb{P}^3$.

The following result on maps of projective spaces and quadrics will be used.

Theorem 2.3. ([16], [6], [8]). *Let* Y *be a smooth projective variety of dimension* n.

(i) *if there exists a dominant regular map* $f : \mathbb{P}^n \to Y$ *then* Y *is isomorphic to* $\mathbb{P}^n$*;*
(ii) *if there exists a dominant regular map* $f : \mathbb{Q}^n \to Y$ *then* $Y \cong \mathbb{P}^n$ *or* $Y \cong \mathbb{Q}^n$ *and in the latter case the map is biregular.*

For the convenience of the reader we recall the following well known result on adjoint bundles which will be used very often throughout the paper.

Theorem 2.4. ([10], (11.2), (11.7), (11.8)) *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 2$. *Let* K *be the canonical bundle on* X. *Then* $K + nL$ *is nef unless* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{Pn}(1))$. *In particular* $K + tL$ *is always nef if* $t > n$. *Suppose that* $K + nL$ *is nef. Then* $K + (n-1)L$ *is nef except in the following cases:*

(i) X *is a hyperquadric in* $\mathbb{P}^{n+1}$ *and* $L = \mathcal{O}_X(1)$*;*

(ii) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*;*

(iii) (X, L) *is a scroll over a smooth curve.*

Suppose that $K + (n-1)L$ *is nef and that* $n > 2$*. Then* $K + (n-2)L$ *is nef except in the following cases:*

(iv) there exists an effective divisor E *on* X *such that* $(E, L_E) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ *and* $[E]_E = \mathcal{O}_E(-1)$*;*

(v0) (X, L) *is a Del Pezzo manifold with* $b_2(X) = 1$*, or* $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(j))$ *with* $j = 2$ *or* 3*,* $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$*, or a hyperquadric in* $\mathbb{P}^4$ *with* $L = \mathcal{O}(2)$*;*

(v1) there is a fibration $f : X \to W$ *over a smooth curve* W *with one of the following properties:*

(v1-V) $(F, L_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for every fiber* F *of* f*;*

(v1-Q) every fiber F *of* f *is an irreducible hyperquadric in* $\mathbb{P}^n$ *having only isolated singularities;*

(v2) (X, L) *is a scroll over a smooth surface.*

3. $\mathbb{P}^1$-bundles as ample divisors.

In this section we will prove a general result about holomorphic $\mathbb{P}^1$-bundles over a smooth projective 3-fold Y with $Y \neq \mathbb{P}^3$ as ample divisor. This will be needed in section 4 to show that some special manifolds cannot be ample divisors in any manifold.

Lemma 3.1. *Let* $g : A \to Y$ *be a holomorphic* $\mathbb{P}^1$*-bundle over a smooth projective manifold* Y *with* $\dim Y = 3$ *and* $Y \neq \mathbb{P}^3$*. Assume that* A *is an ample divisor in a projective manifold* X*. Let* L *be the line bundle on* X *associated to* A*. Then* $K + 4L$ *is nef.*

Proof. Note that X is a 5-dimensional manifold and L is an ample line bundle on X. Using Theorem 2.4 it follows that the bundle $K + 6L$ is always nef and that $K + 5L$ is nef unless $(X, L) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$. But $(X, L) \cong (\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$ implies, being A ample, that $\mathrm{Pic}(A) \cong \mathbb{Z}$ while we know that $\mathrm{Pic}(A) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}$. Hence the bundle $K + 5L$ is nef and again by Theorem 2.4 we see that the exceptions to $K + 4L$ being nef are: $(\mathbb{Q}^5, \mathcal{O}_{\mathbb{Q}^5}(1))$, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, (X, L) is a scroll over a smooth curve B.

The case $(X, L) \cong (\mathbb{Q}^5, \mathcal{O}_{Q^5}(1))$ is ruled out by $\mathrm{Pic}(A) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}$.

The case $(X, L) \cong 4\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is clearly impossible since $\dim X + 5$.

Let (X, L) be a scroll over a smooth curve B. Note that in this case (A, L_A) is a scroll over B. Let $\mathbb{P}^3$ be the general fiber of A over B. Note that $\dim g(\mathbb{P}^3)$ is either 0 or 3. But $\dim g(\mathbb{P}^3) \neq 0$. In fact if $g(\mathbb{P}^3) = y \in Y$ then $\mathbb{P}^3 \subset g^{-1}(y)$.

On the other hand $g^{-1}(y) = \mathbb{P}^1$ since g is a $\mathbb{P}^1$-bundle, hence a contradiction. Thus $\dim g(\mathbb{P}^3) = 3$ and we have a finite surjective morphism from $\mathbb{P}^3$ onto Y. Using Theorem 2.3, we get that $Y \cong \mathbb{P}^3$, which contradicts our assumption. Hence we conclude that $K + 4l$ is nef. $\square$

The following result was claimed by Sommese in ([8], p. 216). In the next proposition we will provide a proof since we don't know any reference for it.

Proposition 3.2. *Let $g : A \to Y$ be a holomorphic $\mathbb{P}^1$-bundle over a smooth projective manifold Y with $\dim Y = 3$ and $Y \neq \mathbb{P}^3$. Assume that $A \neq \mathbb{P}^1 \times \mathbb{P}^3$ and that it is an ample divisor in a projective manifold X. Then $Y \cong \mathbb{Q}^3$.*

Proof. By Lemma 3.1 the adjoint bundle $K + 4L$ is nef. Hence by the Kawamata-Shokurov basepoint free theorem ([15], Sect. 3) there is an integer $k > 0$ such that $|k(K+4L)|$ is base point free. Let $\Phi : X \to W$ be the morphism associated to $|k(K+4L)|$ with k sufficiently large so that $W = \Phi(X)$ is normal and Φ has connected fibers. We have the following possibilities:

(i) $\dim W = 0$ and $K \approx -4L$;
(ii) $\dim W = 1$ and the general fiber of Φ is a smooth quadric $Q \subset \mathbb{P}^5$ with $L_Q \approx \mathcal{O}_Q(1)$;
(iii) $\dim W = 2 < n$, Φ is a $\mathbb{P}^3$-bundle over a smooth surface W and the restriction of L to a fiber is $\mathcal{O}_{\mathbb{P}^3}(1)$;
(iv) $\dim W = n = 5$.

In case (i) the polarized pair (X, L) is a Del Pezzo manifold. Using [12], [14] we see that the Del Pezzo manifolds with $1 \leq d \leq 5$ have $\mathrm{Pic}(X) \cong \mathbb{Z}$ and thus they are ruled out since we know that $\mathrm{Pic}(X) \cong \mathrm{Pic}(A) \cong \mathrm{Pic}(Y) \oplus Z$. The Del Pezzo manifold with degree ≥ 6 cannot occur either since $\dim X = 5$, see ([13], Sect. 5).

Let (X, L) be as in case (iii). Note that in this case (A, L_A) is a scroll over W. Let $\mathbb{P}^2$ be the general fiber of A over W. Note that $\dim g(\mathbb{P}^2)$ is either 0 or 2. But $\dim g(\mathbb{P}^2) \neq 0$. In fact if $g(\mathbb{P}^2) = y \in Y$ then $\mathbb{P}^2 \subset g^{-1}(y)$. On the other hand $g^{-1}(y) = \mathbb{P}^1$ since $g : A \to Y$ is a $\mathbb{P}^1$-bundle. Thus $\dim g(\mathbb{P}^2) = 2$ and we have a finite surjective morphism from $\mathbb{P}^2$ onto $g(\mathbb{P}^2)$. By Theorem 2.3, we see that $g(\mathbb{P}^2) \cong \mathbb{P}^2$. Note that two different $\mathbb{P}^2$'s on Y cannot meet since otherwise the fibers of Φ would intersect. Thus on Y we have a 2-dimensional family of $\mathbb{P}^2$'s and this contradicts $\dim Y = 3$.

Let (X, L) be as in case (iv). Then W is the first reduction of X and $A' = \Phi(A)$ is the first reduction of A. Let $E \cong \mathbb{P}^3$ be an exceptional divisor in A. Note that $\dim g(E) = 3$. Hence we have a finite surjective morphism from $\mathbb{P}^3$ onto Y and using Theorem 2.3, we get that $Y \cong \mathbb{P}^3$, contradicting our assumption.

Let (X, L) be as in case (ii). Then (A, L_A) is also a hyperquadric fibration over W. Let $\mathbb{Q}^3$ be the general fiber of Φ. Note that $\dim g(\mathbb{Q}^3) \neq 0$. In fact if $g(\mathbb{Q}^3) = y \in Y$ then $\mathbb{Q}^3 \subset g^{-1}(y)$. We also have $g^{-1}(y) = \mathbb{P}^1$ since g is a $\mathbb{P}^1$-bundle over Y. Thus $\dim g(\mathbb{Q}^3)$ is either 1, or 2, or 3.

Let $\dim g(\mathbb{Q}^3) = 1$. Let y be a general point in $g(\mathbb{Q}^3)$. Then the general fiber of $g_{|\mathbb{Q}^3} : \mathbb{Q}^3 \to g(\mathbb{Q}^3)$ is 2-dimensional. On the other hand $g^{-1}_{|\mathbb{Q}^3}(y) = g^{-1}(y) \cap \mathbb{Q}^3 = \mathbb{P}^1 \cap \mathbb{Q}^3$. Hence $\dim g^{-1}_{|\mathbb{Q}^3}(y) \leq 1$, a contradiction.

Let $\dim g(\mathbb{Q}^3) = 2$. Let y be a general point in $g(\mathbb{Q}^3)$. Then $\dim g^{-1}_{|\mathbb{Q}^3}(y) = 1$. On the other hand $g^{-1}_{|\mathbb{Q}^3}(y) = g^{-1}(y) \cap \mathbb{Q}^3 = \mathbb{P}^1 \cap \mathbb{Q}^3$. Hence $\mathbb{P}^1 \subset \mathbb{Q}^3$, $g^{-1}_{|\mathbb{Q}^3}(y) = \mathbb{P}^1$ and thus $\Phi(\mathbb{P}^1) = w \in W$. This implies that $(K + 4L)_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}$. We also know that $(K + 4L)_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2) + \mathcal{O}_{\mathbb{P}^1}(4a)$ with $a \geq 1$, a contradiction.

Thus $\dim g(\mathbb{Q}^3) = 3$ and we have a finite surjective morphism from $\mathbb{Q}^3$ onto Y. Since $Y \neq \mathbb{P}^3$, by Theorem 2.3 it follows that $Y \cong \mathbb{Q}^3$. □

4. Fano manifolds of coindex 3 as ample divisors.

In this section we will assume that A is a Fano manifold of coindex 3, $b_2 \geq 2$ and that A is contained as ample divisor in a smooth projective manifold X.

We classify pairs (X, L), where L is the line bundle associated to the divisor A, under the assumption that $\dim A \geq 4$. We are interested in such classification because it is related to the standing conjecture on $\mathbb{P}^d$-bundles ([3], (5.5.1)), for a statement see section 5. In fact among the Fano 4-folds of coindex 3 and $b_2 \geq 2$ there are Fano 4-folds with a $\mathbb{P}^1$-bundle structure either over $\mathbb{P}^3$ or $\mathbb{Q}^3$. We will see that such manifolds are not ample in any manifold X, so that we have examples supporting the conjecture.

Note also that we made the assumption $b_2 \geq 2$ since for $b_2 = 1$ it follows that the Picard number of A is 1 and such pairs (X, A) have been considered in [19].

Proposition 4.1. *Let be a Fano 4-fold of index two and of product type, that is $A \cong \mathbb{P}^1 \times M$, where M is a Fano 3-fold of even index. Assume that A is an ample divisor in a projective manifold X. Then $A \cong \mathbb{P}^1 \times \mathbb{P}^3$ and X is a $\mathbb{P}^4$-bundle over $\mathbb{P}^1$.*

Proof. By Theorem 2.1 it follows that $A \cong \mathbb{P}^1 \times M$, where M is one of the following: $\mathbb{P}^3$, V_d, W, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here V_d is a Del Pezzo manifold with $d = 7$ or $1 \leq d \leq 5$, while W is a divisor of bidegree $(1, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. Thus A can be seen as a $\mathbb{P}^1$-bundle over M. Using Proposition 3.2 we see that if

$M \neq \mathbb{P}^3$ then $M \cong \mathbb{Q}^3$. Thus the only possibility for A is $\mathbb{P}^1 \times \mathbb{P}^3$. Since A is ample in X it follows that X is a $\mathbb{P}^4$-bundle over $\mathbb{P}^1$, see [23]. □

Proposition 4.2. *Let A be a Fano manifold of dimension ≥ 4, coindex 3, $b_2 \geq 2$, with a smooth 3-dimensional section and assume that $A \cong \mathbb{P}^3 \times \mathbb{P}^3$, or $\mathbb{P}^2 \times \mathbb{Q}^3$, or is the blow up of $\mathbb{P}^5$ along a line . Then A cannot be an ample divisor in any manifold.*

Proof. This follows from ([23], Prop. IV) and ([11], (5.8)). □

Proposition 4.3. *Let A be a Fano 4-fold of coindex 3, with $b_2 = 2$ such that A is either the blow up of a smooth 4-dimensional quadric $\mathbb{Q}^4 \subset \mathbb{P}^5$ along a conic C on it such that the plane $< C >$ spanned by C is not contained in $\mathbb{Q}^4$ or the blow up of a smooth 4-dimensional quadric $\mathbb{Q}^4 \subset \mathbb{P}^5$ along a line C. Assume that A is ample in a smooth projective variety X. Then X is the blow up of $\mathbb{P}^5$ along C.*

Proof. For a proof see ([11], (5.10)). □

Proposition 4.4. *Let A be a Fano 4-fold of coindex 3, with $b_2 = 2$ and such that A is a $\mathbb{P}^1$-bundle $P(\mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$ over $\mathbb{P}^3$. Then A cannot be ample in any manifold.*

Proof. Assume that A is an ample divisor in a manifold X. By ([9], (2.1)) it follows that $A \cong \mathbb{P}^1 \times \mathbb{P}^3$, a contradiction. Thus A cannot be ample in any manifold. □

Proposition 4.5. *Let A be a Fano 4-fold of coindex 3, with $b_2 = 2$, such that A has two $\mathbb{P}^1$-bundle structures and can be realized either as $P(NCB)$, where NCB is the null correlation bundle over $\mathbb{P}^3$, or $P(\mathcal{E})$, where $\mathcal{E}$ is a stable rank-2 bundle on $\mathbb{Q}^3$ with $c_1(\mathcal{E}) = -1$, $c_2(\mathcal{E}) = 1$. Then A cannot be an ample divisor in any manifold.*

Proof. Assume that A is an ample divisor in a manifold X. Since A has two $\mathbb{P}^1$-bundle structure we can think of A as a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$. By ([9], (2.1)) it follows that $A \cong \mathbb{P}^1 \times \mathbb{P}^3$, a contradiction. Thus A cannot be ample in any manifold. □

Proposition 4.6. *Let A be a Fano 4-fold of coindex 3, with $b_2 = 2$ and such that A is a $\mathbb{P}^1$-bundle, $P(\mathcal{O}_{\mathbb{Q}^3}(-1) \oplus \mathcal{O}_{\mathbb{Q}^3})$, over $\mathbb{Q}^3 \subset \mathbb{P}^4$. Then A cannot be an ample divisor in any manifold.*

Proof. The idea of the proof is taken from ([9], (2.1)). Let $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(-1) \oplus \mathcal{O}_{\mathbb{Q}^3}$. Assume that A is an ample divisor in a manifold X. Let $p : A \to \mathbb{Q}^3$ be the map which gives to A the structure of a $\mathbb{P}^1$-bundle. We can think of p as the map associated to the linear system $|p^*(\mathcal{O}_{\mathbb{Q}^3}(1))|$. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be the extension of $p^*(\mathcal{O}_{\mathbb{Q}^3}(1))$ to X. Let $F \in |p^*(\mathcal{O}_{\mathbb{Q}^3}(1))|$, i.e. $F = p^{-1}(\mathbb{Q}^2)$. If $\Gamma(X, \mathcal{L}) \to \Gamma(A, Ł_A) \to 0$ then the map p extends to X and this would give the contradiction that $\dim \mathbb{Q}^3 \leq 2$, see ([23], Prop. V). Thus we can assume that $H^1(X, \mathcal{L} - [A]) \neq 0$. This implies that $H^1(A, \mathcal{L}_A - t[A]) \neq 0$ for some $t > 0$. For such t we consider the following exact sequence

$$(1) \qquad 0 \to K_A + t[A] - [F] \to K_A + t[A] \to K_F + t[A]_F - [F]_F \to 0$$

From the cohomology sequence associated to (1), Kodaira vanishing theorem and the fact that $H^3(A, K_A + t[A] - [F]) \neq 0$ since by hypothesis $H^1(A, \mathcal{L}_A - t[A]) \neq 0$, it follows that $H^2(F, K_F + t[A]_F - [F]_F) \neq 0$. Note that F is a $\mathbb{P}^1$-bundle $p_F : F \to \mathbb{Q}^2$. Let $G \in |p^*(\mathcal{O}_{\mathbb{Q}^2}(1))|$, i.e. $G = p_F^{-1}(B)$ where $B \in |p^*(\mathcal{O}_{\mathbb{Q}^2}(1))|$. We consider the sequence

$$(2) \quad 0 \to K_F + t[A]_F - [G] \to K_F + t[A]_F \to K_G + t[A]_G - [G]_G \to 0$$

Reasoning as above we conclude that $H^1(G, K_G + t[A]_G - [G]_G) \neq 0$. This along with the fact that G is a $\mathbb{P}^1$-bundle over $B \cong \mathbb{P}^1$ implies that $G = F_0$. Therefore $\mathcal{E}_B$ is trivial and hence $c_1(\mathcal{E}_B) = 0$. On the other hand $\mathcal{E} = \mathcal{O}_{\mathbb{Q}^3}(-1) \oplus \mathcal{O}^3_{\mathbb{Q}}$ which gives that $c_1(\mathcal{E}_B) \neq 0$, a contradiction. Thus A cannot be ample in any manifold. $\square$

Proposition 4.7. *Let A be a Fano 4-fold of coindex 3 with $b_2 = 2$, such that A is a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$. Then A cannot be an ample divisor in any manifold.*

Proof. Assume that A is an ample divisor in a manifold X. Let L denote the line bundle on X associated to the divisor A. We will show that A cannot be ample in any manifold. The proof will be done in various steps.

Claim 4.8. *X has either two $\mathbb{P}^3$-bundle structures over $\mathbb{P}^2$, or two quadric bundle structures over $\mathbb{P}^2$, or a $\mathbb{P}^3$-bundle structure over $\mathbb{P}^2$ and a quadric bundle structure over $\mathbb{P}^2$.*

Proof of Claim. Since A is a Fano 4-fold of index two there exists an ample divisor H on A such that $2H$ is linearly equivalent to $-K_A$. Going carefully through Sect. 5 in [25] one can see that both extremal rays of A are numerically effective. Let $\phi_1, \phi_2 : A \to \mathbb{P}^2$ be the contraction morphisms of the two rays. By ([25], (1.3)) it follows that no contraction of A has a 3-dimensional fiber. Thus $\phi_1, \phi_2 : A \to \mathbb{P}^2$ are equidimensional with general fiber being a smooth 2-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover H restricted to such fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. Since by assumption the divisor A is ample in X then by ([23], Prop. III) the morphisms ϕ_1, ϕ_2 extend to morphisms $\overline{\phi}_1, \overline{\phi}_2$

from X to $\mathbb{P}^2$. Let $\overline{H}$ be the extension to X of the divisor H, which it exists by the Lefschetz theorem. Note that $\overline{H}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1)$. A reasoning similar to that in ([11], (4.10)) gives that the fiber X_t of $\overline{\phi}_i$, $i = 1, 2$ is either $\mathbb{P}^3$ or a hyperquadric in $\mathbb{P}^4$. In the case in which the general fiber X_t of $\overline{\phi}_i$ is isomorphic to $\mathbb{P}^3$, then $(X, \overline{H}) = (P(E), \xi)$. In the case in which the general fiber of $\overline{\phi}_i$ is isomorphic to a hyperquadric in $\mathbb{P}^4$, then A_t is a hyperplane section of X_t and thus $L_{A_t} = \mathcal{O}_{A_t}(1)$.

We will show that none of the possibilities listed in Claim 4.8 can occur. In order to prove it we need to show that the bundle $K + 3L$ is nef.

Claim 4.9. *$K + 3L$ is nef.*

Proof of Claim. Since $\dim X = 5$ and $\mathrm{Pic}(A) \cong \mathbb{Z} \oplus \mathbb{Z}$, by Theorem 2.4 it follows that $K + tL$ is nef for $t = 6, 5$ and that the exception to $K + 4L$ being nef is: (X, L) is a scroll over a smooth curve B. In this case, since $q(A) = 0$, we have that (A, L_A) is a scroll $(P(E), \xi)$ over $\mathbb{P}^1$. The adjunction formula gives: $K_A = -4\xi + \pi^*(\mathcal{O}_{P^1}(-2) + \det E) = -4L_A + \pi^*\mathcal{O}_{P^1}(e - 2)$, where e is such that $\det E = \mathcal{O}_{\mathbb{P}^1}(e)$. Since -$K_A$ is ample and since (A, L_A) is a scroll over $\mathbb{P}^1$ it follows that $2 - e > 0$. This contradicts the fact that E is an ample rank 4 vector bundle over $\mathbb{P}^1$. Hence the bundle $K + 4L$ is nef and by the Kawamata-Shokurov basepoint free theorem ([15], Sect. 3) there is an integer $k > 0$ such that $|k(K+4L)|$ is base point free. Let $\Phi : X \to W$ be the morphism associated to $|k(K+4L)|$ with k sufficiently large so that $W = \Phi(X)$ is normal and Φ has connected fibers. We have the following possibilities:

(i) $\dim W = 0$ and $K \approx -4L$;
(ii) $\dim W = 1$ and the general fiber of Φ is a smooth quadric $Q \subset \mathbb{P}^5$ with $L_Q \approx \mathcal{O}_Q(1)$;
(iii) $\dim W = 2 < n$, Φ is a $\mathbb{P}^3$ bundle over a smooth surface W and the restriction of L to a fiber is $\mathcal{O}_{\mathbb{P}^3}(1)$;
(iv) $\dim W = n = 5$.

Case (i) cannot occur since this would imply that A is a Fano manifold of index three, contradicting our assumption.

Let (X, L) be as in case (ii). Then (A, L_A) is also a hyperquadric fibration over W. Since we have a morphism from A onto a curve, by ([25], (14)) it follows that $A \cong \mathbb{P}^1 \times M$ where M is either a Fano 3-fold of index two or $\mathbb{P}^3$, a contradiction.

Let (X, L) be as in case (iii). Note that (X, L) is a scroll over W. Let $\mathbb{P}^3$ be the general fiber of X over W. Note that $\overline{\phi}_i(\mathbb{P}^3) = t \in \mathbb{P}^2$. Thus $\mathbb{P}^3 \subset \overline{\phi}_i^{-1}(t)$ which is a quadric in $\mathbb{P}^4$, a contradiction.

Let (X, L) be as in case (iv). Then W is the first reduction of X and $A' = \phi(A)$ is the first reduction of A. Let $E \cong \mathbb{P}^3$ be an exceptional divisor in A. Then $\phi_i(E) = t \in \mathbb{P}^2$ and hence ϕ_i has a 3-dimensional fiber. By ([25], (1.3)) it follows that A is a ruled Fano 4-fold, a contradiction.

We now use Theorem 2.4 to get that $K + 3L$ is nef.

Claim 4.10. *X cannot have two $\mathbb{P}^3$-bundle structure, $\overline{\phi}_i : X \to \mathbb{P}^2$, over $\mathbb{P}^2$, with $i = 1, 2$.*

Proof of Claim. If X has two $\mathbb{P}^3$-bundle structures over $\mathbb{P}^2$, by ([20], Theorem A) it follows that $X \cong \mathbb{P}^2 \times \mathbb{P}^2$, a contradiction.

Claim 4.11. *X cannot have two quadric bundle structures, $\overline{\phi}_i : X \to \mathbb{P}^2$, over $\mathbb{P}^2$, with $i = 1, 2$.*

Proof of Claim. As seen in Claim 4.9, the bundle $K+3L$ is nef. Let $\psi : X \to Y$ be the morphism associated to a sufficiently high power of $K + 3L$. We will prove that the morphism ψ has $\mathbb{P}^2$ as image and moreover that it factors through $\overline{\phi}_i$. In fact since the general fiber Q of $\overline{\phi}_i$ is a hyperquadric in $\mathbb{P}^4$ and since $L_Q = \mathcal{O}_Q(1)$ it follows that $(K + 3L)_Q = \mathcal{O}_Q$. Thus ψ factors through $\overline{\phi}_i$, $\psi = g \circ \overline{\phi}_i$, where $g : \mathbb{P}^2 \to Y$. Note that $Y = \psi(X) = g(\overline{\phi}_i(X)) = g(\mathbb{P}^2)$. Thus $\dim Y = 0, 2$. But $\dim Y = 0$ would imply that (X, A) is a Fano 5-fold of coindex 3 and $b_2 = 2$. Going through the list in ([17], Theorem 6) we see that none of the cases have A as a linear section. Thus $\dim Y = 2$ and hence the morphism $g : \mathbb{P}^2 \to Y$ is onto. Moreover Y is smooth, see [5]. We now use Theorem 2.3 to conclude that $Y \cong \mathbb{P}^2$. We actually have that the morphism g is an isomorphism since it is finite-to-one and since the fibers of both ψ and $\overline{\phi}_i$ are connected. Thus ψ and $\overline{\phi}_i$ are the same (modulo g). And hence the same holds for ϕ_1, ϕ_2. But this is impossible since ϕ_i are two different contractions. Thus we conclude that X cannot have two quadric bundle structures.

Claim 4.12. *X cannot have a $\mathbb{P}^3$-bundle structure, $\overline{\phi}_1 : X \to \mathbb{P}^2$, over $\mathbb{P}^2$ and quadric bundle structure, $\overline{\phi}_2 : X \to \mathbb{P}^2$, over $\mathbb{P}^2$.*

Proof of Claim. Assume that $\overline{\phi}_1 : X \to \mathbb{P}^2$ is a $\mathbb{P}^3$-bundle and that $\overline{\phi}_2 : X \to \mathbb{P}^2$ is a quadric bundle. Let $\mathbb{P}^3$ be a general fiber of $\overline{\phi}_1$. Note that $\overline{\phi}_2(\mathbb{P}^3) = t \in \mathbb{P}^2$. Thus $\mathbb{P}^3 \subset \overline{\phi}_2^{-1}(t)$ which is a quadric in $\mathbb{P}^4$, a contradiction.

Thus we have shown that A cannot be an ample divisor in any manifold X. $\square$

Proposition 4.13. *Let A be a Fano 4-fold of coindex 3 with $b_2 = 2$ and such that A is a divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 2)$. Assume that A is ample in a manifold X. Then either X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^3$, or X is a non* *equidimensional scroll over a normal 3-fold, or X is a quadric bundle over $\mathbb{P}^2$.*

Proof. Let H be an ample divisor on A such $2H$ is linearly equivalent to $-K_A$. Going carefully through Sect. 5 in [25] one can see that both extremal rays of A are numerically effective. Let ϕ_1 and ϕ_2 be the two contraction morphisms. Under our assumption, we see that $\phi_1 : A \to \mathbb{P}^2$ and $\phi_2 : A \to \mathbb{P}^3$. Moreover by ([25], (1.3)) it follows that no contraction of A has a 3-dimensional fiber. Thus $\phi_1 : A \to \mathbb{P}^2$ is equidimensional with general fiber being a smooth 2-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover H restricted to such fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. As for $\phi_2 : A \to \mathbb{P}^3$, such morphism has a finite number of fibers of dimension 2, each one of them being isomorphic to $\mathbb{P}^2$, see ([25], (1.2) along with (5.1)). Now since the divisor A is ample in X, by ([23], Prop. III) the morphism ϕ_1 extends to a morphism $\overline{\phi}_1$ from X to $\mathbb{P}^2$. Let $\overline{H}$ be the extension to X of the divisor H, which it exists by the Lefschetz theorem. Note that $\overline{H}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. A reasoning similar to that in ([12], (4.10)) gives that the fiber X_t of $\overline{\phi}_1$ is either $\mathbb{P}^3$ or a hyperquadric in $\mathbb{P}^4$. In the former case $(X, \overline{H}) = (P_{\mathbb{P}^2}(E), \xi)$ in the latter case A_t is a hyperplane section of X_t and thus $L_{A_t} = \mathcal{O}_{A_t}(1)$.

Let $(X, \overline{H}) = (P_{\mathbb{P}^2}(E), xi)$. We use adjunction theory to understand the structure of the polarized pair (X, L). We show first that the bundle $K + 3L$ is nef, where $L = \mathcal{O}_X(A)$. The proof is essentially the same as that in Claim 4.9. The only case which has to be treated differently is the one in which the morphism $\Phi : X \to W$ associated to a sufficiently high power of $K + 4L$ has a 2-dimensional image. Note that the general fiber $\mathbb{P}^3$ of Φ is sent via $\overline{\phi}_1$ to a point since $\dim \overline{\phi}_1(\mathbb{P}^3)$ can be either 0 or 3. Thus we get a morphism $g : \mathbb{P}^2 \to W$ such that $g \circ \overline{\phi}_1 = \Phi$. Moreover the morphism g is onto and therefore by Theorem 2.3 we get that $W \cong \mathbb{P}^2$. In order to see that such case cannot occur we argue as follows. The morphism $g : \mathbb{P}^2 \to W$ is finite to one and since the fibers of both Φ and $\overline{\phi}_1$ are connected it follows that g is indeed an isomorphism. This implies that Φ and $\overline{\phi}_1$ are the same (modulo g) and hence $\phi_1 = \Phi_A$. But $\Phi_A : A \to \mathbb{P}^2$ is a scroll over $\mathbb{P}^2$, a contradiction since we know that the general fiber of ϕ_1 is $\mathbb{P}^1 \times \mathbb{P}^1$. Thus we conclude, using Theorem 2.4, that $K + 3L$ is nef. Let $\psi : X \to Y$ be the morphism with connected fibers and normal image Y associated to a sufficiently high power of $K + 3L$. Note that $\dim Y \leq 3$ or $\dim Y = 5$, see [3] or [10].

If $\dim Y = 0$ then (X, L) is a Fano 5-fold of coindex 3 and $b_2 = 2$. Going through the list in ([17], Theorem 6) we see that none of the cases have A as a linear section.

If $\dim Y = 1$ then A has a morphism, ψ_A, onto a smooth curve Y and by ([25], (1.4)) A must be a ruled Fano manifold, a contradiction.

If $\dim Y = 2$ then $\psi : X \to Y$ is a quadric bundle over Y. Let $\mathbb{P}^3$ be a general fiber of $\overline{\phi}_1$. Note that $\psi(\mathbb{P}^3) = t \in Y$. Thus $\mathbb{P}^3 \subset \psi^{-1}(t)$ which is a quadric in $\mathbb{P}^4$, a contradiction.

If $\dim Y = 3$ then the general fiber F of ψ is such that $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. We will consider separately the following two cases:

(a) ψ is equidimensional;
(b) otherwise.

In case (a), since ψ is equidimensional, by ([13], (2.12)) it follows that Y is smooth and that $(X, L) = (P_Y(\mathcal{E}), \xi^{\mathcal{E}})$, where $\xi^{\mathcal{E}}$ is the tautological line bundle of $\mathcal{E}$. Note that the fiber $\mathbb{P}^3$ of $\overline{\phi}_1$ cannot be sent to a point via ψ, since the fibers of ψ are $\mathbb{P}^2$'s. Thus $\psi(\mathbb{P}^3) = Y$ and, by Theorem 2.3, $Y \cong \mathbb{P}^3$. Hence our manifold X has two projective bundle structures: one over $\mathbb{P}^2$ and the other one over $\mathbb{P}^3$. By ([20], Theorem A) we get that $X \cong \mathbb{P}^2 \times \mathbb{P}^3$. It is easy to see that $L_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$ and $L_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(2)$ and thus A is a divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ of type $(1, 2)$.

In case (b) X is a non equidimensional scroll over a normal 3-fold Y.

If $\dim Y = 5$ then $\psi : X \to Y$ is birational. Note that $\psi_A : A \to Y$ is also birational and it contracts some curves. By ([4], (0.4.3)) there exists an extremal rational curve C such that $(K_A + 2L_A) \cdot C = 0$. Moreover by ([4], (0.7)), one can choose an extremal rational curve l such that $R = \mathbb{R}_+[l]$, $(K_A + 2L_A) \cdot l = 0$ and $-K_A \cdot l = \text{length}(R)$. Let f be the contraction morphism associated to R. Then ψ_A factors through f, i.e. $\psi_A = g \circ f$. Hence in particular f is birational. Thus R is not nef. This is a contradiction since we are in the case in which both extremal rays of A are nef.

We now consider the case in which the fiber of $\overline{\phi}_1$ is isomorphic to a (possibly singular) hyperquadric in $\mathbb{P}^4$. In this case A_t is a hyperplane section of X_t and thus $[A]_{A_t} = \mathcal{O}_{A_t}(1)$. Reasoning as in the proof of Claim 4.11, we conclude that the morphism ψ associated to a high power of $K + 3L$ has $\mathbb{P}^2$ as image. Thus $\psi : X \to \mathbb{P}^2$ is a quadric bundle. Moreover such ψ factor through $\overline{\phi}_1$. $\square$

Proposition 4.14. *Let A be a Fano 4-fold of coindex* 3 *with* $b_2 = 2$ *and such that A is a divisor of* $\mathbb{P}^2 \times \mathbb{Q}^3$ *of bidegree* $(1, 1)$*. Assume that A is ample in a manifold X. Then either X is isomorphic to* $\mathbb{P}^2 \times \mathbb{Q}^3$*, or X is isomorphic to* $\mathbb{P}^2 \times \mathbb{P}^3$ *or X is a quadric bundle over* $\mathbb{P}^2$*, or X is a non equidimensional scroll over a normal 3-fold Y.*

Proof. Let H be an ample divisor on A such that $2H$ is linearly equivalent to $-K_A$. Going carefully through Sect. 5 in [25] one can see that both extremal rays of A are numerically effective. Let ϕ_1 and ϕ_2 be the two contraction morphisms. Under our assumption, we see that $\phi_1 : A \to \mathbb{P}^2$ and $\phi_2 : A \to \mathbb{Q}^3$. Moreover by ([25], (1.3)) it follows that no contraction of A has a 3-dimensional fiber. Thus $\phi_1 : A \to \mathbb{P}^2$ is equidimensional with general fiber being a smooth 2-dimensional quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover H restricted to such fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ (1,1). As for $\phi_2 : A \to \mathbb{Q}^3$, such morphism has a finite number of fibers of dimension 2 and each one of them is isomorphic to $\mathbb{P}^2$, see ([25], (1.2) along with (5.1)). Now since the divisor A is ample in X, by ([23], Prop. III) the morphism ϕ_1 extends to a morphism $\overline{\phi}_1$ from X to $\mathbb{P}^2$. Let $\overline{H}$ be the extension to X of the divisor H, which it exists by the Lefschetz theorem. Note that $\overline{H}_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. A reasoning similar to that in ([12], (4.10)) gives that the fiber X_t of $\overline{\phi}_1$ is either $\mathbb{P}^3$ or a hyperquadric in $\mathbb{P}^4$. In the former case $(X, \overline{H}) = (P_{\mathbb{P}^2}(E), \xi)$ in the latter case A_t is a hyperplane section of X_t and thus $L_{A_t} = \mathcal{O}_{A_t}(1)$.

If $(X, \overline{H}) = (P_{\mathbb{P}^2}(E), \xi)$, a reasoning similar to the corresponding case in Proposition 4.13 gives that $K+3L$ is nef. Let $\psi : X \to Y$ be the morphism with connected fibers and normal image Y associated to a sufficiently high power of $K + 3L$. Note that $\dim Y \leq 3$ or $\dim Y = 5$, see [3], [10].

If $\dim Y = 0$ then (X, L) is a Fano 5-fold of coindex 3 and $b_2 = 2$. Going through the list in ([17], Theorem 6) we see that $X \cong \mathbb{P}^2 \times \mathbb{Q}^3$.

If $\dim Y = 1, 2, 5$ we rule these cases out as in Proposition 4.13.

If $\dim Y = 3$ then the general fiber F of ψ is such that $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and the following two cases occur:

(a) ψ is equidimensional;
(b) otherwise.

Reasoning as in Proposition 4.13, we get that either $X \cong \mathbb{P}^2 \times \mathbb{P}^3$ and A is a divisor in $\mathbb{P}^2 \times \mathbb{P}^3$ of type $(1, 2)$, or X is a non equidimensional scroll over a normal 3-fold Y.

If the fiber of $\overline{\phi}_1$ is isomorphic to a (possibly singular) hyperquadric in $\mathbb{P}^4$ then, as in Proposition 4.13, we get that X is a quadric bundle over $\mathbb{P}^2$. □

Proposition 4.15. *Let A be a Fano 4-fold of coindex 3 with $b_2 = 2$ and such that A is the intersection of two divisors of bidegree $(1, 1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$. Assume that A is ample in a manifold X. Then either X is isomorphic to a divisor of bidegree $(1, 1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$, or $X \cong \mathbb{P}^2 \times \mathbb{Q}^3$, or X is a quadric bundle over $\mathbb{P}^2$, or X is a scroll over a smooth 3-fold Y, or X is a non equidimensional scroll over a normal 3-fold Y.*

Proof. Since the fourfold A is the intersection of two divisor of bidegree (1, 1 on $\mathbb{P}^3 \times \mathbb{P}^3$, going carefully through Sect. 5 in [25] one can see that both extremal rays of A are numerically effective. Let $\phi_1, \phi_2 : A \to \mathbb{P}^3$ be the contraction morphisms of the two rays. Since A is not a ruled Fano manifold, by ([25], (1.2) and (5.1)) it follows that there exists a fiber of ϕ_i isomorphic to $\mathbb{P}^2$.

In order to understand the structure of the manifold X containing A as an ample divisor we use adjuction theory. We start by showing that the adjoint bundle $K + 3L$ is nef, where $L = \mathcal{O}_X(A)$. The proof is essentially as in Claim 4.9. The cases to be treated differently are the one in which the morphism $\Phi : X \to W$ associated to a sufficiently high power of $K + 4L$ has 2-dimensional image and the case in which Φ is birational.

We consider first the case in which the morphism $\Phi : X \to W$ has a 2-dimensional image. In this case $\Phi_A : A \to W$ is a scroll over W. Let $\mathbb{P}^2$ be a 2-dimensional fiber of ϕ_i. Note that such $\mathbb{P}^2$ is rigid and thus its image via Φ_A is W. Thus we have an onto morphism $\Phi_{\mathbb{P}^2} : \mathbb{P}^2 \to W$ and using Theorem 2.3, we get that $W \cong \mathbb{P}^2$. Thus (A, L_A) is a scroll $(P(E), \xi)$ over $\mathbb{P}^2$. By the adjunction formula we get that $-K_A = 3L_A + \pi^*(\mathcal{O}_{\mathbb{P}^2}(3 - c_1(E)))$, where we take L_A and $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ as generators of Pic(A). On the other hand, since A is a Fano manifold of index 2, it follows that $-K_A = 2H$ for some ample line bundle H on A. The line bundle H, with respect to the basis L_A and $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, will be of the form $H = \alpha L_A + \beta\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ for some $\alpha, \beta \in \mathbb{Z}$. Combining the latter with the adjuction formula we get that $3L_A + \pi^*(\mathcal{O}_{\mathbb{P}^2}(3 - c_1(E))) = 2\alpha L_A + 2\beta\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, which gives $3 = 2\alpha$ and $3 - c_1(E) = 2\beta$, a contradiction since $\alpha, \beta \in Z$.

We consider next the case in which the morphism $\Phi : X \to W$ is birational. Note that the restriction morphism $\Phi_A : A \to \Phi_A(A)$ is the morphism associated to some high power of $K_A + 3L_A$. Such morphism is also birational and it contracts some curves. By ([4], (0.4.3)) there exists an extremal rational curve C such that $(K_A + 3L_A) \cdot C + 0$. Moreover by ([4], (0.7)), one can choose an extremal rational curve l such that $R = \mathbb{R}_+[l]$, $(K_A + 2L_A \cdot l = 0$ and $-K_A \cdot l = \text{length}(R)$. Let f be the contraction morphism associated to R. Then Φ_A factors through f, i.e. $\Phi_A = g \circ f$. Hence in particular f birational. Thus R is not nef and this is impossible since, as we have remarked earlier, in this case both extremal rays of A are nef. Thus we can conclude, using Theorem 2.4, that the adjoint bundle $K + 3L$ is nef.

Let $\psi : X \to Y$ be the morphism with connected fibers and normal image Y associated to a sufficiently high power of $K + 3L$. Using [3], [10] we have that $\dim Y \leq 3$ or $\dim Y = 5$.

If $\dim Y = 0$ then (X, L) is a Fano 5-fold of coindex 3 and $b_2 = 2$. Going through the list in [([17], Theorem 6) we see that either X is a divisor of bidegree (1,1) on $\mathbb{P}^3 \times \mathbb{P}^3$, or $X \cong \mathbb{P}^2 \times \mathbb{Q}^3$.

If $\dim Y = 1$ then A has a morphism, ψ_A, onto a smooth curve and by ([25], (1.4)) A must be a ruled Fano manifold, a contradiction.

If $\dim Y = 2$ then $\psi : X \to Y$ is a quadric bundle over Y. Note that $\psi_A : A \to Y$ is also a quadric bundle over Y. Let $\mathbb{P}^2$ be a 2-dimensional fiber of ϕ_i. Note that $\mathbb{P}^2$ cannot be sent to a point via ψ_A since ψ_A is a quadric bundle. Thus $\psi_A(\mathbb{P}^2) = Y$. Moreover Y is smooth, see [5]. Using Theorem 2.3 we get that $Y \cong \mathbb{P}^2$.

If $\dim Y = 3$ then the general fiber F of ψ is such that $(F, L_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and the following two cases occur:

(a) ψ is equidimensional;
(b) otherwise.

In case (a), since ψ is equidimensional, by ([13], (2.12)) it follows that Y is smooth and that $(X, L) = (P_Y(\mathcal{E}), \xi)$, where ξ is the tautological bundle of $\mathcal{E}$. In case (b) X is a non equidimensional scroll over a normal 3-fold Y.

If $\dim Y = 5$ then $K + 3L$ is nef and big and thus $K_A + 2L_A$ is nef and big. As we have seen earlier, the fact that $K_A + 2L_A$ is nef and big would imply the existence in A of a not nef extremal ray. This latter fact is not possible since both extremal rays of A are nef. $\square$

5. Remarks.

For the convenience of the reader we recall the standing conjecture on smooth $\mathbb{P}^d$-bundles, ([3], (5.5.1)), and see how the manifolds considered are natural candidates for examples supporting such conjecture.

Conjecture 5.1. ([3], (5.5.1)). *Let L be an ample line bundle on a smooth projective variety, X, of dimension $n \geq 3$. Assume that there is a smooth $A \in |L|$ such that A is a $\mathbb{P}^d$-bundle, $p : A \to B$, over a manifold B, of dimension b. Then $d \geq b - 1$ and it follows that $(X, L) \cong (P(\mathcal{E}), H)$, for an ample vector bundle, $\mathcal{E}$, on B with p equal to the restriction to A of the induced projection $P(\mathcal{E}) \to B$, except if either:*

(i) $X \subset \mathbb{P}^4$ *is a quadric and* $L \cong \mathcal{O}_{\mathbb{P}^4}(1)_X$,
(ii) $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$;
(iii) $A \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$, *$p$ is the product projection onto the second factor, $(X, L) \cong (P(\mathcal{E}), H)$, for an ample vector bundle, $\mathcal{E}$, on $\mathbb{P}^1$ with the product projection of A onto the first factor equal to the induced projection* $P(\mathcal{E}) \to \mathbb{P}^1$.

Remark 5.2. The Fano manifold $\mathbb{P}^1 \times \mathbb{Q}^{n-1}$ cannot be an ample divisor in any manifold if $n \geq 3$. In fact T. Fujita has proved that if A is a fiber bundle over a manifold S with fiber being a smooth hyperquadric in $\mathbb{P}^n$, then A cannot be ample divisor in any manifold if $n \geq 3$, see ([13], (4.10)). Hence, in particular, the Fano manifold $\mathbb{P}^1 \times \mathbb{Q}^{n-1}$ cannot be an ample divisor in any manifold if $n \geq 3$.

On passing we would like to point out that this was not noted in ([19], (2.5)) and consequently (2.6) in [19] is not precise.

Remark 5.3. The Fano manifold $\mathbb{P}^1 \times \mathbb{Q}^3$ can be seen as a $\mathbb{P}^1$-bundle over $\mathbb{Q}^3$ and, as remarked earlier, it cannot be ample in any manifold.

The manifold $\mathbb{P}^1 \times \mathbb{Q}^3$ is certainly an example supporting the above conjecture. The other ones are those discussed in 4.4, 4.5, 4.6.

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